Math 104 004 **Final Exam** **May 13, 2022**

Name: ____

SID: ____

Instructions :

1. You have 170 minutes, 7:10pm-10:00pm. You may not need that much time.
2. No books, notes, or other outside materials are allowed.
3. There are 9 questions on the exam. Each question is worth either 5 or 10 points, for a total of 75 points.
4. You need to show all of your work and justify all statements, unless otherwise noted. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

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Total	

1. a) (3 points) Prove, directly from (either) definition, that $[0, 1]$ is a closed set.

   b) (3 points) Prove, directly from (either) definition, that $[0, 1]$ is not an open set.

   c) (4 points) Let $t_n = \sin(\frac{n\pi}{4})$ for all $n \in \mathbb{N}$. Prove that (t_n) does not converge.

a) Let (s_n) be a sequence in $[0,1]$ which converges to $t \in \mathbb{R}$. Since $0 \leq s_n \leq 1$ for all $n \in \mathbb{N}$, $0 \leq t \leq 1$ and $t \in [0,1]$. So $[0,1]$ is closed.

b) Let $\varepsilon > 0$. $0 \in [0,1]$, but $(-\varepsilon, \varepsilon)$ contains negative numbers like $-\frac{\varepsilon}{2}$, so $(-\varepsilon, \varepsilon) \not\subseteq [0,1]$ for any $\varepsilon > 0$. Thus $[0,1]$ is not open.

c) Consider the subsequences

$t_K = s_{4K} = \sin(K\pi) = 0$, $t_K \to 0$

$r_K = s_{8K+2} = \sin(2K\pi + \frac{\pi}{2}) = 1$, $r_K \to 1$

Since the limits are different (s_n) does not converge

2. (5 points) Let $s_n = \frac{n+4}{n^3-5}$ for all $n \in \mathbb{N}$. Find $s \in \mathbb{R}$ such that $s_n \to s$. Prove, directly from the definition, that $s_n \to s$.

$s = 0$.

Let $\varepsilon > 0$. Choose $N = \sqrt{\frac{5}{\varepsilon} + 5}$

If $n \in \mathbb{N}$, $n > N$ then $n > 1$ so

$$\left| \frac{n+4}{n^3-5} \right| = \frac{n+4}{n^3-5} \le \frac{n+4n}{n^3-5n} = \frac{5}{n^2-5}$$

$$< \frac{5}{N^2-5} = \varepsilon .$$

3. a) (5 points) Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{\sqrt{x}}$. Prove, directly from the definition, that f is uniformly continuous.

b) (5 points) Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = \frac{1}{\sqrt{x}}$. Prove that g is not uniformly continuous.

a) Let $\varepsilon > 0$. Choose $\delta = 2\varepsilon$

If $x, y \in [1, \infty)$, $|x-y| < \delta$, then

$$\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}} \right| = \left| \frac{\sqrt{x} - \sqrt{y}}{\sqrt{xy}} \right| = \left| \frac{x-y}{\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})} \right| < \frac{\delta}{2} = \varepsilon$$

Sice $\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y}) \geq (1+1) = 2$.

b) $(\frac{1}{n^2})$ is a sequence in $(0, \infty)$ which is Cauchy (indeed, it converges to 0).

But $g(\frac{1}{n^2}) = n$ is not bounded and thus not Cauchy. So g is not uniformly continuous.

4. (5 pts) Let $f, g : [1,2] \to \mathbb{R}$ be continuous functions which are differentiable on $(1,2)$. Suppose $f(1) = -2$, $g(1) = 1$, $f'(x) \geq 3$ and $g'(x) \leq -2$ for all $x \in (1,2)$. Prove that there exists $x \in [1,2]$ such that $f(x) = g(x)$.

Define $h : [1,2] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$.

Then h is continuous and diff on $(1,2)$.

By MVT, there exists $x \in (1,2)$ such that

$$\frac{h(2) - h(1)}{2-1} = h'(x) = f'(x) - g'(x) \geq 5$$

$h(1) = f(1) - g(1) = -3$, so

$h(2) \geq h(1) + 5 = 2$. Since 0 is between -3 and 2, the IVT implies there exists $y \in (1,2)$ such that

$h(y) = f(y) - g(y) = 0$.

5. (10 points) Prove, directly from the definition of the integral, that if $f, g : [a, b] \to \mathbb{R}$ are integrable functions such that $f(x) \leq g(x)$ for all $x \in [a, b]$ then $\int_a^b f \leq \int_a^b g$.

Let $S \subseteq [a,b]$. For all $x \in S$

$$m(f,S) \leq f(x) \leq g(x) \leq M(g,S)$$

$$M(f,S) \leq M(g,S) \quad \text{and}$$
$$m(f,S) \leq m(g,S)$$

It follows that for any partition P of $[a,b]$

$$\int_a^b f = U(f) \leq U(f,P) \leq U(g,P)$$

So $\int_a^b f$ is a lower bound for

$\{U(g,P) \mid \text{partions } P \text{ of } [a,b]\}$ and

$$\int_a^b f \leq U(g) = \int_a^b g.$$

6. (5 points) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that there exist $a, b \in \mathbb{R}$ such that $f([0,1]) = [a,b]$.

Since $[0,1]$ is compact and f cont.
(because closed and bounded)

$f([0,1])$ is compact and thus closed and bounded. Let $a = \inf f([0,1])$ $b = \sup f([0,1])$. There are sequences in $f([0,1])$ converging to a and b, so by def. of closed set $a, b \in f([0,1])$

Since $[0,1]$ is connected (an interval) $f([a,b])$ is an interval, and contains a and b. So all numbers between a and b and

$[a,b] \subseteq f([0,1]) \subseteq [a,b]$ and

$f([0,1]) = [a,b]$.

7. Let $f : (0, \infty) \to \mathbb{R}$ be $f(x) = \frac{1}{x}$.

a) (2 points) f is integrable on any interval $[a, b] \subset (0, \infty)$. Why?

b) (3 points) Define $F : (0, \infty) \to \mathbb{R}$ by $F(x) = \int_1^x f$. Prove that F is differentiable and find F'.

c) (2 points) Prove that F is an increasing function.

d) (3 points) Let $G : \mathbb{R} \to \mathbb{R}$ be a differentiable function such $G(x) > 0$ for all $x \in \mathbb{R}$ and $F \circ G(x) = x$. Use the chain rule to prove that $G'(x) = G(x)$ for all $x \in \mathbb{R}$.

a) f is continuous, so integrable.

b) Since f is continuous, F is diff. by the FTC part II and $F' = f$.

c) $(0, \infty)$ is an interval and $F' = f > 0$ on $(0, \infty)$, so F is increasing

d) $1 = (x)' = (F \circ G)'(x) = F'(G(x))\, G'(x)$

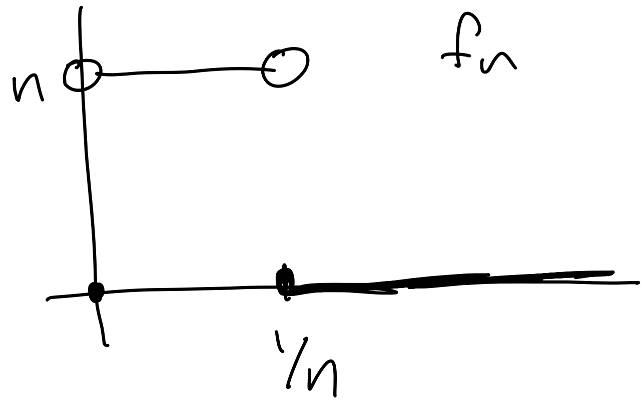
$= f(G(x))\, G'(x) = \dfrac{G'(x)}{G(x)} = 1.$

8. For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n$ for all $x \in (0, \frac{1}{n})$, and $f_n(x) = 0$ for all $x \in [0,1] \setminus (0, \frac{1}{n})$.

   a) (5 points) Find a function $f : [0,1] \to \mathbb{R}$ such that $f_n \to f$ pointwise. Justify your answer.

   b) (5 points) Does $f_n \to f$ uniformly? Justify your answer.

a)



$f(x) = 0$ for all $x \in [0,1]$.

If $x = 0$, $f_n(0) = 0 \to 0 = f(0)$.

If $x > 0$, for all $n > \frac{1}{x}$ $f_n(x) = 0$, so $f_n(x) \to 0 = f(x)$.

b) No. Choose $\varepsilon = 1$. Let $N \in \mathbb{R}$. If $n \in \mathbb{N}$, $n > N$, then $\frac{1}{2n} \in (0, \frac{1}{n}) \subset [0,1]$, but $|f_n(\frac{1}{2n}) - f(\frac{1}{2n})| = |n - 0| = n \geq 1 = \varepsilon$.

9. Consider the power series $\sum_{n=1}^{\infty}(n+(-1)^n)6^n x^n$.

a) (7 points) Find an interval $I \subseteq \mathbb{R}$ such that the power series converges uniformly to a function $f : I \to \mathbb{R}$.

b) (3 points) Find a power series converging to $f' : I \to \mathbb{R}$.

a) $|(n+(-1)^n)6^n|^{1/n} \leq |2n6^n|^{1/n} = 6(2n)^{1/n} \to 6$

So $\limsup |(n+(-1)^n)6^n|^{1/n} \leq 6$

and $R \geq \frac{1}{6}$. Thus the power series converges uniformly on $[-\frac{1}{7}, \frac{1}{7}]$.

b) $\sum_{n=1}^{\infty} n(n+(-1)^n)6^n x^{n-1}$.